Statistics and Medieval Astronomical Tables

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1. Introduction

Around the year 150 A.D. the Greek scientist Claudius Ptolemy, who lived and worked in Alexandria in present-day Egypt, compiled his main astronomical work, the *Almagest*. In this work he not only collected the most important achievements of his predecessors, in particular Hipparchus (Rhodes, second century B.C.), but he also developed the first accurate geometrical models for the motion of the moon and the five visible planets as seen from the earth. Ptolemy computed large sets of tables, mostly of complicated functions based on plane or spherical trigonometry, by means of which the geocentric positions of the sun, moon and planets could be calculated at the cost of only a small number of additions and multiplications.

Ptolemy's astronomical work was highly influential till the end of the Middle Ages. From the ninth till the sixteenth century Muslim astronomers from Afghanistan till Spain and from Yemen till Constantinople (see also figure 1) used the *Almagest* as a prototype for their own astronomical handbooks with tables, which were mainly written in Arabic or Persian. Only incidentally they made modifications to the Ptolemaic planetary models, but in many cases they increased the accuracy of the calculations underlying the tables, in particular by determining more accurate values for the basic trigonometric functions. Furthermore, they made new observations of the motions of the sun, moon and planets, on the basis of which they improved upon the values of the parameters underlying the models. In some

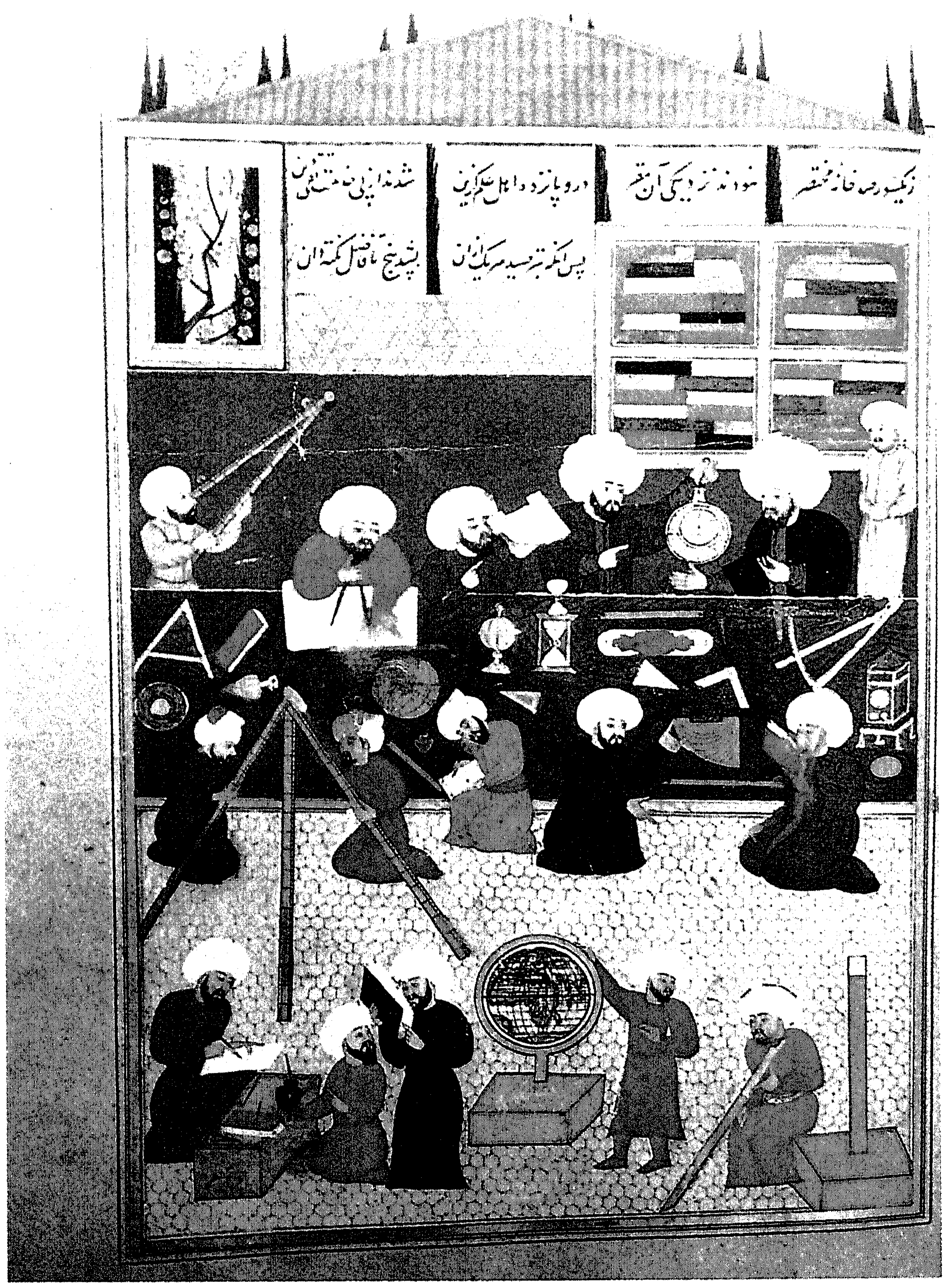

Figure 1. Astronomers at work in the observatory of Constantinople (16th century).

cases, e.g., the eccentricity of the solar orbit or the obliquity of the ecliptic (the angle between the equator and the plane in which the apparent yearly motion of the sun takes place), these values had actually changed during the centuries. In other cases, e.g. the length of the solar year, the availability of observations over longer time spans allowed a more accurate determination.

Frequently the extant manuscripts of medieval astronomical handbooks are quite different from the original works. Since the manuscripts were copied by hand, many scribal errors occurred, in particular in the numbers in the tables. Often we find additions to the text or the tables by later users and in many cases parts from other works were inserted in place of missing, outdated or unsuitable parts of the original work. Knowledge of the origin of such added materials could give us important information about the development of medieval astronomy and the influence of earlier astronomers on later ones. However, in many cases the added material is not properly attributed.

A useful method for determining from which sources tables in an astronomical handbook derive is the comparison of mathematical properties of the tables, e.g. the underlying function and parameter values, pecularities of the method of computation such as intermediate rounding, the use of (inverse) interpolation or inaccurate auxiliary tables, etc. This kind of information can sometimes be found in tabular headings or explanatory text, but is often unreliable or simply missing. Not always is a given table based on the indicated parameter values and only incidentally leads a recomputation following the rules in the explanatory text precisely to the tabular values. Sometimes tables are based on two different values for the same astronomical parameter.

From the above it follows that for many tables in medieval astronomical handbooks methods by which the mathematical properties can be determined directly from the tabular values are indispensable. Until recently such methods were only applied *ad hoc*: tables were recomputed for various historically attested parameter values and methods of computation in order to see which value and method led to the best agreement, the use of linear interpolation was recognized from groups of constant tabular differences, etc. Although many important results were found in this way, there remained a large class of tables which defied a mathematical explanation. Only in the last decade has a systematic approach to the analysis of medieval astronomical tables with the use of advanced mathematics and statistics and special computer programs been practiced by a small number of scholars.

2. The SMC research project in Utrecht

In a SMC-supported research project at the Mathematical Institute of Utrecht University an extensive investigation of the application of statistical methods for the analysis of medieval astronomical tables was made. Firstly, the conditions under which statistics can be applied to such tables, i.e. under which errors in tables can be assumed to be random variables, were explored (see below). Secondly, several estimators were developed for determining unknown parameter values underlying medieval astronomical tables. Since such tables mostly display values for complicated functions based on plane or spherical trigonometry, the estimators are non-trivial. They include:

- **Weighted estimator.** Assume that we have a table with values $T(x)$ for a function f_θ depending on a single parameter θ. Let g be a function of two variables such that $g(x, f_\theta(x)) = \theta$ for every argument x and every value of θ. The value of θ underlying the table can then be estimated from each single tabular value $T(x)$ using $\theta \approx g(x, T(x))$. Bias and variance of these estimators can be approximated using a Taylor expansion of g. Finally, a more accurate estimator is obtained by computing a weighted sum of the separate estimators.
- **Fourier Estimator.** Let g be a 2π-periodic odd function and c an unknown constant. Let f be defined by $f(x) = g(x - c)$ for every x. If the Fourier series of f converges and the Fourier coefficients a_k and b_k for $k > 0$ are given by

  $$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \qquad \text{and} \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

  respectively, we have $a_k \cos kc + b_k \sin kc = 0$ for every k. From a table for f we can calculate approximations $\hat{a}_k$ and $\hat{b}_k$ to the Fourier coefficients by replacing the integral by a finite sum and the (unknown) functional values by tabular values. Then $\tan c$ can be estimated by the quotients $-\hat{a}_k/\hat{b}_k$ (or $\cot c$ by $-\hat{b}_k/\hat{a}_k$). These estimators have some interesting properties, e.g. they become degenerate if c lies precisely between two consecutive arguments of the table for f.
- **Least Number of Errors Criterion.** According to this criterion an unknown parameter value underlying an astronomical table is determined in such a way that the number of errors in the table is as small as possible. The criterion can be given a statistical interpretation if we assume a probability distribution for the tabular errors.
- **Least Squares Estimator.** The method of least squares was used for determining multiple unknown parameters from a single table.

The accuracy of the estimators listed above was determined and confidence intervals for the unknown parameters computed. Special user-friendly computer programs were written to deal with the sexagesimal number system (i.e. with base 60) in which values in medieval astronomical tables were usually given. In these programs the estimators described above and many specific methods of analysis can be conveniently applied.

The SMC research project in Utrecht led to a large number of interesting historical results which had not been possible without the use of statistical methods and special computer programs. One of these was the recovery of some of the lost tables of the important tenth-century Muslim astronomer Abū'l-Wafā᾿ in a thirteenth-century astronomical handbook (see below). Furthermore, the method of computation of various tables for complicated functions which had thus far defied explanation, was discovered.

3. Can statistics be applied to medieval astronomical tables?

In order to apply statistical methods to ancient and medieval astronomical tables, we assume that, in some sense, the tabular values behave as random variables. We will investigate this in the case of three spherical-astronomical tables by Abū'l-Wafā᾿. This important Muslim mathematician and astronomer lived and worked in tenth-century Baghdad and is known in particular for the advances he made in spherical astronomy and in the calculation of accurate sine values. Abū'l-Wafā᾿'s major astronomical work, the *Almagest*, has a structure similar to Ptolemy's *Almagest*. Most of the explanatory text concerning trigonometry, spherical astronomy and the planetary models is extant in a manuscript in the Bibliothèque Nationale in Paris. The text includes geometrical proofs, directions for the calculation of many useful quantities in spherical astronomy, and a large number of numerical examples. However, the tables belonging to the original work, explicitly indicated to follow the explanatory sections, are not present; a possible reason is that the person who ordered the copying of the manuscript, was only interested in the mathematics, not in the tables.

The Bibliothèque Nationale also possesses a unique manuscript of the astronomical handbook of al-Baghdādī (thirteenth century), which is a mixture of material from earlier works. al-Baghdādī explicitly states that he used some of the lunar tables of Ḥabash al-Ḥāsib (Baghdad, c. 840) and the solar equation table of Abū'l-Wafā᾿. Furthermore, he copied some inaccurate tables for spherical astronomy from Kūshyār ibn Labbān, a contemporary of Abū'l-Wafā᾿. al-Baghdādī also included a more accurate set of tables for trigonometry and spherical astronomy. By comparing the mathematical properties of these tables with the explanatory text in Abū'l-Wafā᾿'s *Almagest*, it is possible to show that the tables derive from that work.

We will now investigate the tabular errors in three of Abū'l-Wafā᾿'s tables occurring in the astronomical handbook of al-Baghdādī. In each case $T(x)$ denotes the tabular value for argument x and the tabular error $e(x)$ is defined as the difference between $T(x)$ and the exact functional value $f(x)$: $e(x) = T(x) - f(x)$. A tabular value is called correct if it is equal to $f(x)$ rounded to the number of digits of the table concerned; otherwise, the tabular value is said to contain an error. All tabular values considered below are given in sexagesimal notation, i.e. to base 60. In transcriptions of sexagesimal numbers a semicolon denotes the sexagesimal point and further sexagesimal digits are separated by commas. For instance, the sexagesimal number 51;57,41,29 denotes $51 + 57 \cdot 60^{-1} + 41 \cdot 60^{-2} + 29 \cdot 60^{-3}$ and will be said to have three sexagesimal fractional digits or to be given to sexagesimal thirds.

Example 1: Abū'l-Wafā''s sine table
Like many ancient and medieval astronomers, Abū'l-Wafā' tabulated the trigonometric functions for a base circle with radius 60. Thus his sine table displayed values for $\mathrm{Sin}\, x \stackrel{\mathrm{def}}{=} 60 \cdot \sin x$. These values were given to sexagesimal thirds (e.g., $T(60) = 51;57,41,29$), which corresponds to an accuracy of approximately seven decimal digits. Abū'l-Wafā''s original table is known to have had values for each quarter of a degree, but the copy which is extant in the handbook of al-Baghdādī only displays values for each integer degree. Most of the ninety tabular values are correct; only nine values contain an error of ± 1 sexagesimal third. Figure 2 displays the tabular errors $e(x) \stackrel{\mathrm{def}}{=} T(x) - \mathrm{Sin}\, x$ in Abū'l-Wafā''s sine table for all arguments $x = 1, 2, 3, \ldots, 90$. The nine tabular errors outside the range from −0;0,0,0,30 to +0;0,0,0,30 correspond to the nine errors just mentioned. All tabular errors within the range correspond to correct tabular values; they are precisely the errors made in rounding exact sine values to the number of digits of the table. It appears that these rounding errors behave like independent random variables with a uniform distribution on the interval [−0;0,0,0,30 , +0;0,0,0,30]. Below this idea will be further pursued.

Example 2: Abū'l-Wafā''s tangent table
Abū'l-Wafā''s original tangent table displayed values for $\mathrm{Tan}\, x \stackrel{\mathrm{def}}{=} 60 \cdot \tan x$ for each quarter of a degree; the values for arguments up to 45° had four sexagesimal fractional digits (e.g., $T(30) = 34;38,27,39,38$), those for arguments larger than 45°, three (e.g., $T(60) = 103;55,22,58$). Again in his version of the table al-Baghdādī left out the tabular values for non-integer arguments.

The tabular errors $e(x) \stackrel{\mathrm{def}}{=} T(x) - \mathrm{Tan}\, x$ in the tangent table do not display a uniform pattern like that in the sine table. Instead they contain a clear tendency towards larger errors for larger arguments. This pattern can be explained as follows:
medieval astronomer usually calculated the tangent from his previously calculated sine values according to

$$\mathrm{Tan}\, x = \frac{60 \cdot \mathrm{Sin}\, x}{\mathrm{Sin}\,(90 - x)}.$$

If he used sine values from a sine table with a fixed number of sexagesimal fractional digits, the relative error in the denominator increased as x approached 90°. Since the error in the numerator remained of the same order,

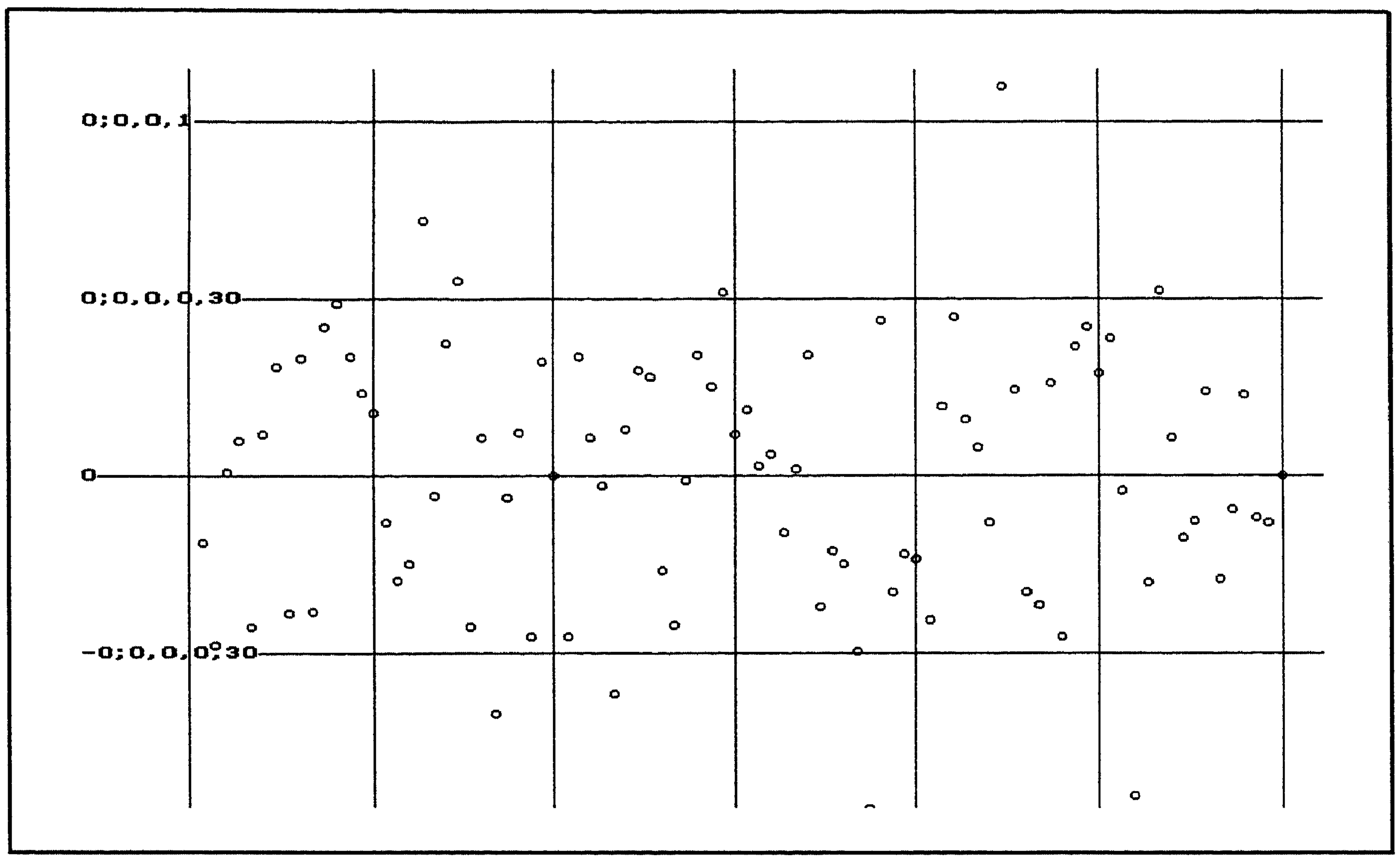

Figure 2. Plot of the tabular errors in Abū'l-Wafā'’s sine table.

the absolute error in the quotient increased. This is precisely what happens with Abū'l-Wafā'’s tangent table.

Figure 3 shows the differences between the tangent values presented by al-Baghdādī and values recomputed from his sine table as described above. The increasing tendency towards 90° has completely disappeared; instead, both in the first half of the table (although this is difficult to see from figure 3) and in the second half the differences appear to behave as independent random variables. Although the calculation of tangent values from the sine table as described above is a relatively easy and in principle completely deterministic process, we note that the differences shown in figure 3 apparently contain two sources of randomness: firstly, the random error in the sine values; secondly, a more or less random error made in the division of the sine values: apparently the quotient was not calculated to its full accuracy and the last digit was to a certain extent guessed. This led to the 15 differences larger than 0;0,0,0,0,30 for arguments between 0° and 45° and to the 13 differences larger than 0;0,0,0,30 for arguments between 45° and 90° which can be seen in figure 3.

It can be checked that changes in the underlying sine values of only one sexagesimal third lead to clearly different tangent values. Therefore we can be certain that the tangent table in the astronomical handbook of al-Baghdādī was computed from his sine table. The outlying difference for argument 41° is probably due to a scribal or computational error.

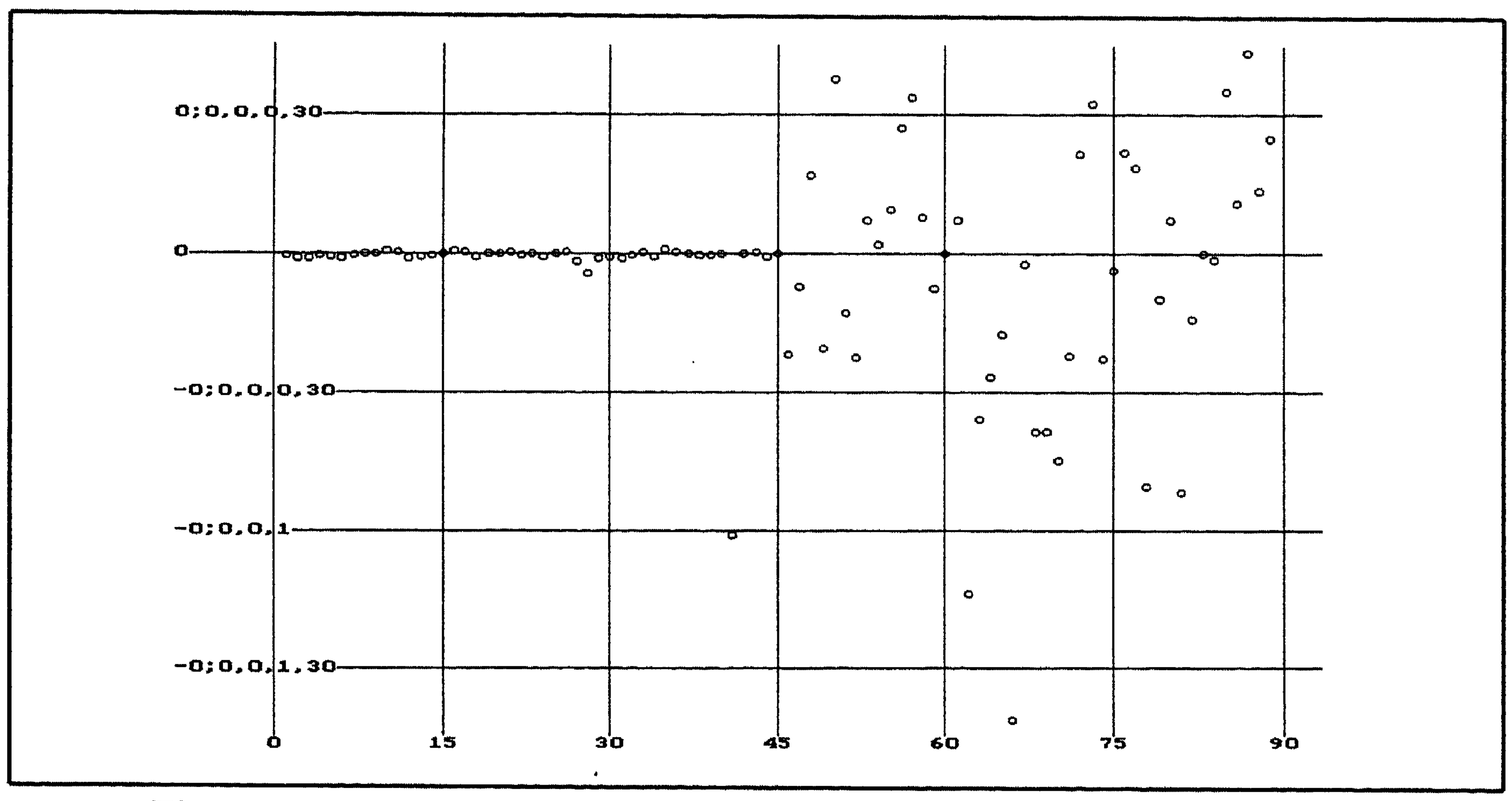

Figure 3. Plot of differences between tangent table and reconstruction.

Example 3: Abū'l-Wafā''s solar declination table
The solar declination is the orthogonal distance on the heavenly sphere between the sun and the equator. It can be calculated by applying the sine law to the spherical triangle ΥSP in figure 4, where E denotes the earth, s the sun, λ the solar longitude (i.e., the length of the arc ΥS along the ecliptic), ε the obliquity of the ecliptic, and δ the declination. We have

$$\delta(\lambda) = \arcsin(\sin\varepsilon \cdot \sin\lambda)$$

for each value of λ. Practically every medieval astronomical handbook contained tables for the solar declination and other spherical astronomical functions connected to the transformation of equatorial coordinates into

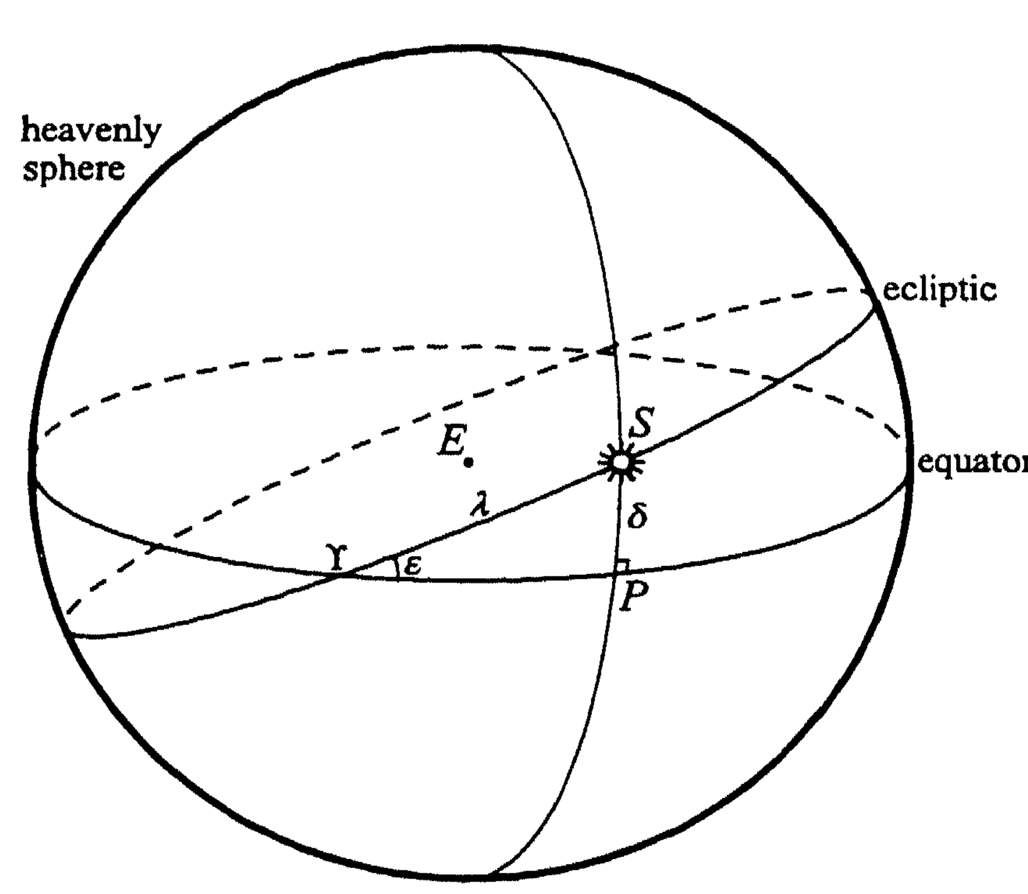

Figure 4. The calculation of the solar declination.

ecliptical ones and vice versa, or to the determination of the time of the day from observations of the sun or a fixed star.

The tabular errors in Abū'l-Wafā''s table for the solar declination, which gives values to sexagesimal thirds for arguments $1, 2, 3, \ldots, 90$, do not display a uniform pattern like the ones above. Instead the errors tend to be negative rather than positive and they are up to twenty times as large as the level of rounding of the table (here we disregard the apparent outlier for argument 79°). An explanation can only be found by following the calculation of the tabular values step by step. For the calculation of the solar declination according to the formula above, Abū'l-Wafā' needed:

1) sine values for arguments $1, 2, 3, \ldots, 90$,
2) a value for $\sin \varepsilon$, and
3) a method to calculate an arcsine.

Naturally he could take the required sine values directly from his table of sines. Furthermore, he could calculate $\sin \varepsilon$ by performing interpolation in his sine table or by making a separate, more accurate, but highly elaborate calculation. Finally, he had to determine the arcsines by performing some type of inverse interpolation in his sine table. In his *Almagest* Abū'l-Wafā' consistently used the value 23;35 for the obliquity of the ecliptic and 24;0,17,38 for $\mathrm{Sin}\,\varepsilon$. This value can be obtained by performing a linear interpolation between the accurate values 23;55,29,48 for Sin 23;30 and 24;9,53,17 for Sin 23;45 (as we have seen above, Abū'l-Wafā''s original sine table displayed values for each quarter of a degree). Abū'l-Wafā' also described how to determine the arc corresponding to a given sine value, namely by means of inverse linear interpolation in his sine table. (Second order interpolation schemes were also known around the year 1000 A.D., but were not widely used for the computation of tables.)

It can be checked that Abū'l-Wafā''s table for the solar declination was indeed computed using the value 24;0,17,38 for $\sin \varepsilon$ and inverse linear interpolation between accurate sine values for each quarter of a degree. If we calculate the differences between Abū'l-Wafā''s declination values and values reconstructed according to this method of computation, we note the following:

- Of the 90 differences seven have an absolute value larger than 4 sexagesimal thirds. Several of these outliers are close to round numbers (5, 10 or 40 thirds), which makes it plausible that they were caused by scribal or computational errors at some stage of the calculation. The absolute value of three more differences is larger than $1\frac{1}{2}$ thirds.

- 53 differences lie between −0;0,0,0,30 and +0;0,0,0,30 and hence correspond to declination values correctly calculated and rounded according to the method of computation described above. These differences appear to behave like independent uniformly distributed random variables.

- The remaining 27 differences have an absolute value between 0;0,0,0,30 and 0;0,0,1,30 and hence correspond to declination values in whose calculation small errors were made. It appears that the behaviour of these errors can be considered to be random. The following causes of the errors can be conjectured:
    1. For non-integer arguments we have used correct sine values to sexagesimal thirds because Abū'l-Wafā''s complete original table is no longer extant. However, since the values for integer arguments contain nine errors, it is probable that also the values for non-integer arguments contain errors. For example, the three errors in the reconstructed declination values for arguments 81 to 83 disappear if we assume an error of one unit in one of the underlying sine values.
    2. The product 24;0,17,38 $\cdot \sin\lambda$ was probably not calculated to its full accuracy. It might have been rounded or some of the smaller terms of the product may simply have been left out.
- The differences are much larger in number and in size if inverse interpolation within smaller or larger intervals of the sine table is used. Apparently, the error pattern in the table is highly characteristic for the intervals used for the interpolation.

Both in the case of the tangent and in the case of the solar declination we have seen that if we take into account the systematic causes of error in the tabular values and leave out the outliers caused by unpredictable scribal or computational errors, we are left with errors that seem to behave like independent random variables with distributions that can be assumed to be equal for practical purposes. We will now discuss the distribution of the tabular errors in some more detail.

4. The probability distribution of tabular errors

We have seen that the errors in medieval astronomical tables consist of the following components:

1. Scribal and computational errors, which are unpredictable but can sometimes be corrected on the basis of the form of the numerals used or by reconstructing the consecutive steps in the calculation of the tabular values.
2. Systematic errors, which are due to the specific method of computation of the tabular values (cf. the tangent and solar declination tables discussed above).

3. Random errors, which can be considered to be independent and identically distributed. For a table with hardly any errors, this component is constituted by the rounding errors (cf. the correct values in the sine table discused above); below we will argue that in many practical cases the rounding errors can be considered to be independent and uniformly distributed. Small non-systematic errors in a table (as we have found in the tangent and solar declination tables) can be assumed to be the result of inaccuracies in the steps of the calculation: rounding of intermediate results, use of auxiliary tables with errors, use of interpolation in auxiliary tables, etc. In this case the tabular errors could be assumed to have a uniform distribution on the interval corresponding to the level of rounding of the tabular values and 'normal tails' outside that interval.

4.1. Distribution of rounding errors

In particular in Abū'l-Wafā''s sine and tangent tables we have seen that the rounding errors seemed to behave like random variables with a uniform distribution. We will now argue that under certain conditions rounding errors in a correct table of an astronomical function have approximately a uniform distribution and can be assumed to be independent. First we note that the values produced by the linear congruential random number generator $x_{k+1} = (ax_k + c) \bmod m$ are rounding errors of a table with equidistant arguments and unit m for an exponential function $f(x) = ba^x + d$, where the constants b and d depend on a, c and the initial value x_0 of the sequence of random numbers. It seems reasonable to expect that under certain conditions, in particular if the tabulated function is not (almost) linear and if the number of sexagesimal fractional digits of the tabular values is sufficiently large, the rounding errors in tables occurring in ancient and medieval astronomical handbooks have approximately a uniform probability distribution and are independent. Thus we can for instance conjecture the following:

Let $T_{k,n}$ be a correct table with k sexagesimal fractional digits for the non-linear function f, such that $T_{k,n}$ has n tabular values for equidistant arguments x_i, $i = 1, 2, 3, \ldots, n$ in a fixed interval. For every argument x_i the tabular error $e_{k,n}(x_i)$ defined by $e_{k,n}(x_i) = T_{k,n}(x_i) - f(x_i)$ is the rounding error that we make by rounding the exact functional value $f(x)$ to the number of sexagesimal digits of the table. I will assume that this rounding is performed in the modern way. Let $F_{k,n}$ be the experimental distribution of the normalized rounding errors of the table $T_{k,n}$, i.e., $F_{k,n}(y)$ is the fraction of rounding errors smaller than $60^{-k}y$ for every y. Note that we have $F_{k,n}(y) = 0$ for every $y \leq -\frac{1}{2}$ and $F_{k,n}(y) = 1$ for every $y \geq +\frac{1}{2}$. Let F_k be the limiting distribution of $F_{k,n}$ for $n \to \infty$.

Conjecture. F_k converges to the uniform distribution on $[-\frac{1}{2}, +\frac{1}{2}]$ as k tends to infinity.

For certain types of functions it may be possible to prove this conjecture by means of the theory of *uniform distribution modulo one*. Let $\{x\}$ denote the fractional part of the real number x. For a given sequence x_i, $i = 1, 2, 3, \ldots$ of real numbers let $A_M(a, b)$ be the number of terms x_i, $1 \leq i \leq M$, for which $\{x_i\} \in [a, b\rangle$. The sequence x_i, $i = 1, 2, 3, \ldots$ is said to be uniformly distributed modulo one if

$$\lim_{M \to \infty} \frac{A_M(a, b)}{M} = b - a \quad \text{for every } 0 \leq a < b \leq 1. \tag{4.1}$$

Using the notation introduced above, our conjecture can now be stated as follows: the sequence $\frac{1}{2} + f(x_i) \cdot 60^k$, $i = 1, 2, 3, \ldots$ is asymptotically uniformly distributed modulo one for $k \to \infty$.

Less general results can be obtained if we introduce randomness explicitly. By means of an unpublished theorem by J.H.B. Kemperman it can be shown that for a randomly chosen argument the distribution of the rounding error converges to the uniform distribution as the number of sexagesimal fractional digits tends to infinity. Since the roles of the argument and an underlying parameter can be interchanged, the same holds for a randomly chosen value of an underlying parameter.

We expect that the rounding errors in a table for a non-linear function will become independent as the number of sexagesimal fractional digits tends to infinity. For instance, we may conjecture that the joint experimental distribution of rounding errors for arguments with fixed distances converges to the joint distribution of independent uniform variables as the number of sexagesimal fractional digits of the tabular values approaches infinity.

REFERENCES

1. B. VAN DALEN (1993). *Ancient and Mediaeval Astronomical Tables: mathematical structure and parameter values*, Ph.D. Thesis, Utrecht University.

2. E.S. KENNEDY ET AL. (1983). *Studies in the Islamic Exact Sciences*, Beirut.

3. L. KUIPERS, H. NIEDERREITER (1974). *Uniform Distribution of Sequences*, New York.

4. O. PEDERSEN (1974). *A Survey of the Almagest*, Odense.

5. G.R. VAN BRUMMELEN (1993). *Mathematical Tables in Ptolemy's Almagest*, Ph.D. Thesis, Simon Fraser University, Burnaby B.C.